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Strong Local Optimality Conditions for State Constrained Control Problems

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Abstract. In this paper first- and second-order optimality conditions for strong local minimum are presented for optimal control problems with pure state set-inclusion constraints. The first-order condition is of Pontryagin type, while the second-order condition is of the form of an accessory problem associated with the strong local minimality. This latter condition contains an extra term reflecting the presence of the pure state constraints.

Key words: Critical cone, Critical tangent cone, First- and second-order strong local optimality conditions, Set-valued constraints

1. Introduction

Consider the optimal control problem

$$(\mathcal{P}) \qquad \text{Minimize}\,\ell(x(0),x(1))\,\text{subject to}$$

$$\dot{x}(t)=f(t,x(t),u(t)) \text{ for a.e. } t\in[0,1], \tag{1.1}$$

$$b(x(0),x(1))=0, \tag{1.2}$$

$$k(t,x(t))\in S \text{ for } t\in[0,1], \tag{1.3}$$

$$u(t)\in U \text{ for a.e. } t\in[0,1], \tag{1.4}$$

where $x:[0,1]\to\mathbb{R}^n$ is absolutely continuous, $u:[0,1]\to\mathbb{R}^m$ is measurable, S is a convex subsets of $\mathbb{R}^\kappa$ with nonempty interior, and U is an arbitrary subset of $\mathbb{R}^m$.

If a pair (x,u) satisfies (1.1)—(1.4), then it is said to be an *admissible* process for $\mathcal{P}$.

The study of second-order *weak* local optimality conditions for the problem $\mathcal{P}$ in [10] – [11] had revealed that the presence of the pure-state constraints (1.3) produces an extra term evoking the following notions. Namely, the set of second-order admissible variations and its support functional, and the critical cone.

However, for the case when the pure state-constraint is absent, Dubovitskii and Milyutin had introduced in [1] a time transformation that transforms the problem $\mathcal{P}$ into a problem $(\mathcal{AP})$ with the property that the optimality in $\mathcal{P}$ yields the optimality in $(\mathcal{AP})$ of the transformed solution. Furthermore, when the first-order *weak* local optimality conditions are applied to $(\mathcal{AP})$ and then carefully analyzed, one deduces the Pontryagin principle for the problem $\mathcal{P}$.

The main task of this paper is to present not only *first-* but also *second*-order optimality conditions for *strong* local minimum in $\mathcal{P}$ when the pure state constraints are present. The method we employ is a serious modification of the time transformation provided in [1] (see also [3]).

The paper is divided as follows. The next section consists of listing the notions and results from earlier papers and which are needed for this paper. The main result of this paper is presented in section 3. An outline of the proof of the main result is furnished also in Section 3. For some details of the proof see [12].

2. Notations and Terminology

First we recall from [4] some notions and notations concerning first- and second-order differentiability properties of functions between normed spaces.

Let Z, Y be normed spaces, D be an open subset of Z and $H:D\to Y$. We say that H is strictly Fréchet differentiable at a point $\widehat{z}\in D$ if there exists a bounded linear operator $H'(\widehat{z}):Z\to Y$ with the property that, for all $\eta>0$, there exists $\delta>0$ such that

$$\|H(z')-H(z'')-H'(\widehat{z})(z'-z'')\|\leqslant\eta\|z'-z''\|$$

whenever z' and z'' satisfy $\|z'-\widehat{z}\|<\delta$ and $\|z''-\widehat{z}\|<\delta$. If the above inequality is required only for $z''=\widehat{z}$, then F is called *Fréchet differentiable at* $\widehat{z}$.

If H is Fréchet differentiable at $\widehat{z}$ and $d\in Z$, then we say that H is *twice directionally differentiable at* $\widehat{z}$ *in direction* d if the following limit exists

$$H''(\widehat{z};d):=\lim_{s\to 0+}2\frac{H(\widehat{z}+sd)-H(\widehat{z})-sH'(\widehat{z})d}{s^2}.$$

Clearly, the above limit always exists if $d=0$.

If in addition, T is a topological space, $H:D\times T\to Y$ and $\widehat{z}:T\to D$ is a continuous function, then we say that $H(\cdot,t)$ *is uniformly strictly Fréchet differentiable at* $\widehat{z}(t)$ *for* $t\in T$ if, for each fixed $t\in T$, there exists a linear operator $H'(\widehat{z}(t),t):Z\to Y$ such that, for all $\eta>0$, there exists $\delta>0$ such that, for all $t\in T$,

$$\|H(z',t)-H(z'',t)-H'(\widehat{z}(t),t)(z'-z'')\|\leqslant\eta\|z'-z''\|$$

whenever z' and z'' satisfy $\|z'-\widehat{z}(t)\|<\delta$ and $\|z''-\widehat{z}(t)\|<\delta$.

If $H(\cdot,t)$ is Fréchet differentiable at $\widehat{z}(t)$ and $d:T\to Z$, is a continuous function, then we say that $H(\cdot,t)$ is *twice uniformly directionally differentiable at $\widehat{z}(t)$ in direction $d(t)$ for $t\in T$* if, for all $t\in T$, the following limit exists

$$H''(\widehat{z}(t),t;d(t)) := \lim_{s\to 0+} 2\frac{H(\widehat{z}(t)+sd(t),t)-H(\widehat{z}(t),t)-sH'(\widehat{z}(t),t)d(t)}{s^2}$$

and the convergence is uniform in $t\in T$, that is, for all $\eta>0$ there exists $\delta>0$ such that, for all $t\in T$ and for all $0<s<\delta$,

$$\left| H''(\widehat{z}(t),t;d(t)) - 2\frac{H(\widehat{z}(t)+sd(t),t)-H(\widehat{z}(t),t)-sH'(\widehat{z}(t),t)d(t)}{s^2}\right| < \eta.$$

The notion of second-order admissible variation set used to express second-order necessary conditions (defined first by Dubovitskii and Milyutin in [2]) is described in the following way.

Let Z be a normed space, $\mathscr{S}\subset Z$, $z\in\mathscr{S}$, and $d\in Z$. A vector $v\in Z$ is called a *second-order admissible variation of $\mathscr{S}$ at z in the direction d* if there exists $\overline{\varepsilon}>0$ such that

$$z+sd+s^2(v+u)\in\mathscr{S} \qquad \text{for all} \quad 0<s<\overline{\varepsilon},\ \|u\|<\overline{\varepsilon},\ u\in Z.$$

The set of all such variations is denoted by $V(z,d|\mathscr{S})$. It follows directly from the definition that $V(z,d|\mathscr{S})$ is an open set. If $\mathscr{S}$ is also convex, then $V(z,d|\mathscr{S})$ is convex as well.

In order to derive meaningful second-order optimality conditions, it is required to select directions d that guarantee the nonemptiness of $V(z,d|\mathscr{S})$. Such directions $d\in Z$ are labeled as the *critical directions of $\mathscr{S}$ at z* and form a set called *critical direction cone to $\mathscr{S}$ at z*. Throughout this paper, this cone will be denoted by $C(z|\mathscr{S})$. It can be easily seen that $C(z|\mathscr{S})$ is a *convex cone* if $\mathscr{S}$ is convex.

The following result offers a characterization of the critical cone (cf. [10]).

THEOREM 2.1. *Let Z be a normed space, $\mathscr{S}\subset Z$ be a closed convex set with nonempty interior. Let $z\in\mathscr{S}$ and $d\in Z$. Then $d\in C(z|\mathscr{S})$ if and only if there exists a constant $M>0$ such that*

$$\langle\zeta,d\rangle^2 \leqslant M\|\zeta\|[\delta^*(\zeta|\mathscr{S})-\langle\zeta,z\rangle] \quad \textit{for all } \zeta\in Z^* \textit{ satisfying } \langle\zeta,d\rangle>0. \tag{2.1}$$

The condition formulated in Theorem 2.1 can be expressed equivalently in the following way: *There exists a constant $M>0$ such that*

$$-E(z,d|\mathscr{S})(\zeta) \leqslant \frac{M\|\zeta\|}{4} \qquad \textit{for all} \quad \zeta\in Z^* \quad \textit{satisfying } \langle\zeta,d\rangle>0,$$

where $E(z,d|\mathscr{S})(\zeta) := \frac{\langle\zeta,d\rangle^2}{4[\langle\zeta,z\rangle-\delta^*(\zeta|\mathscr{S})]}$ *for $\zeta\in Z^*$ such that $\zeta\notin N(z|\mathscr{S})$. Here δ^* denotes the support functional.* □

In the special case when Z is finite dimensional we can explicitely express the support function of $V(z,d|\mathcal{S})$. For, we need the following notations. If $\mathcal{S}=S\subset\mathbb{R}^\kappa$, $z\in S$ and $d\in\mathbb{R}^\kappa$ then denote

$$d^\perp:=\{\zeta\in\mathbb{R}^\kappa\,|\,\langle\zeta,d\rangle=0\},\qquad d^>:=\{\zeta\in\mathbb{R}^\kappa\,|\,\langle\zeta,d\rangle>0\},$$

and define from $\mathbb{R}^\kappa$ to the extended reals the function

$$\mathscr{E}(z,d|S)(\zeta):=\begin{cases}\liminf\limits_{\substack{\xi\to\zeta\\ \xi\in d^>}} E(z,d|S)(\xi), & \text{if } \zeta\in N(z|S)\cap d^\perp,\\ \quad +\infty, & \text{otherwise.}\end{cases}$$

One can see that $\mathscr{E}(z,d|S)(\cdot)$ is a positively homogeneous function and also lower semicontinuous on $\mathbb{R}^\kappa\setminus\{0\}$. Define the convex regularization $\overline{\mathrm{co}}\mathscr{E}(z,d|S)(\cdot)$ to be the largest lower semicontinuous convex function below $\mathscr{E}(z,d|S)(\cdot)$, that is,

$$\overline{\mathrm{co}}\mathscr{E}(z,d|S)(\zeta) = \sup\{\ \varphi(\zeta)\ |\ \varphi:\mathbb{R}^\kappa\to[-\infty,\infty] \text{ is convex and lsc,}$$
$$\varphi(\xi)\leqslant\mathscr{E}(z,d|S)(\xi)\ \forall\xi\in\mathbb{R}^\kappa\setminus\{0\}\ \}.$$

It results that $\overline{\mathrm{co}}\mathscr{E}(z,d|S)(\cdot)$ is also sublinear. The following result offers an evaluation of the support function of the set $V(z,d|S)$ (cf. [9, Corollary 2.7]).

THEOREM 2.2. *Let S be a closed convex set with nonempty interior and $d\in C(z|S)$, and let $\zeta\in\mathbb{R}^\kappa$. Then*

$$\delta^*\big(\zeta\big|V(z,d|S)\big)=\overline{\mathrm{co}}\mathscr{E}(z,d|S)(\zeta).$$

Consider now the special case when $X=\mathscr{C}(T,\mathbb{R}^\kappa)$, where $T=(T,\rho)$ is a compact metric space, and $S\subset\mathbb{R}^\kappa$ is closed and convex set with nonempty interior. Define the $\mathcal{S}\subset\mathscr{C}(T,\mathbb{R}^\kappa)$ as the set of continuous selections of S by

$$\mathcal{S}=\mathrm{sel}_C(S):=\{z\in\mathscr{C}(T,\mathbb{R}^\kappa)\,|\,z(t)\in S \text{ for } t\in T\}.$$

Then $\mathcal{S}$ is closed convex set of $\mathscr{C}(T,\mathbb{R}^\kappa)$.

Regarding $\mathcal{S}=\mathrm{sel}_C(S)$, a thorough study of convex analysis concepts (normal and tangent cones, support function, etc.) was developed in [6]. Results concerning the second-order admissible variations, critical cone, and application to abstract optimization were derived in [5]. A far simpler characterization of the set of critical directions is offered by the following results from [5, Theorem 3.5, Lemmas 3.6, 3.8].

THEOREM 2.3. *Let $\mathcal{S}=\mathrm{sel}_C(S)$, $z\in\mathcal{S}$. Then $d\in\mathscr{C}(T,\mathbb{R}^\kappa)$ is in the critical cone $C(z|\mathcal{S})$ if and only if there exists a constant $M>0$ such that, for all $t\in T$,*

$$\langle\zeta,d(t)\rangle^2\leqslant M\|\zeta\|(\delta^*(\zeta|S)-\langle\zeta,z(t)\rangle)\quad \textit{whenever}\quad \zeta\in\mathbb{R}^\kappa \textit{ and } \langle\zeta,d(t)\rangle>0. \tag{2.2}$$

Since the dual space of $\mathscr{C}(T,\mathbb{R}^\kappa)$ consists of signed Borel measures due to the Riesz representation theorem, then the characterization $C(z|\mathscr{S})$ obtained via Theorem 2.1 requires verifying (2.1) for measures ζ. On the other hand, (2.2) needs to be verified for $\zeta\in\mathbb{R}^\kappa$. This fact renders Theorem 2.3 more valuable in applications. Another consequence of Theorem 2.3 is concerning the connection between $C(z|\mathscr{S})$ and the set-valued mapping $t\mapsto C(z(t)|S)$. From Theorem 2.1 applied to $\mathscr{S}:=S$ (where t is kept fixed), it results, that $d(t)\in C(z(t)|S)$ is equivalent to the fact that (2.2) holds for some constant $M_t>0$. Therefore, Theorem 2.3 can be reformulated as follows:
A continuous function d belongs to $C(z|\mathscr{S})$ if and only if

$$d(t)\in C(z(t)|S) \qquad (t\in T), \tag{2.3}$$

and the corresponding constants M_t from (2.2) can be chosen to be uniformly bounded.

When (2.2) is valid for some constant M and for all $t\in T$, then we say that (2.3) holds uniformly in $t\in T$. The result analogous to Theorem 2.2 for the case of $\mathscr{S}=\mathrm{sel}_C(S)$ requires more involved notions. Let $z\in\mathrm{sel}_C(S)$ and $d\in C(z|\mathrm{sel}_C(S))$. Denote by $d^\#:T\to 2^{\mathbb{R}^\kappa}$ the following set-valued function

$$d^\#(t)=\{\,\zeta\in\mathbb{R}^\kappa\,|\,\exists t_n\to t,\ \exists\zeta_n\to\zeta \text{ with } \zeta_n\in d(t_n)^>\ \forall n\in\mathbb{N}\,\}.$$

Define

$$\mathbb{E}(z,d|S)(t,\zeta):=\begin{cases}\displaystyle\liminf_{\substack{(s,\xi)\to(t,\zeta)\\ \xi\in d(s)^>}} E(z(s),d(s)|S)(\xi), & \text{if } \zeta\in N(z(t)|S)\cap d(t)^\perp\cap d^\#(t),\\ 0, & \text{if } \zeta\in N(z(t)|S)\cap d(t)^\perp\setminus d^\#(t),\\ +\infty, & \text{otherwise,}\end{cases} \tag{2.4}$$

Define the convex regularization $\overline{\mathrm{co}}\,\mathbb{E}(z,d|S)(\cdot,\cdot)$ to be the largest lower semicontinuous function $\varphi:T\times\mathbb{R}^\kappa\to[-\infty,\infty]$ below $\mathbb{E}(z,d|S)(\cdot,\cdot)$ such that, for each $t\in T$, the function $\zeta\mapsto\varphi(t,\zeta)$ is convex on $\mathbb{R}^\kappa$.

In the following result, we describe how the support functional of $V(z,\,d|\mathrm{sel}_C(S))$ can be evaluated in terms of $\overline{\mathrm{co}}\,\mathbb{E}$.

THEOREM 2.4. *Let T be a compact metric space, $S\subset\mathbb{R}^\kappa$ be closed and convex with nonempty interior. Let $z\in\mathrm{sel}_C(S)$, $d\in C(z|\mathrm{sel}_C(S))$ and let μ be a bounded vector-valued Borel measure on T. Then*

$$\delta^*\big(\mu\big|V(z,d|\mathrm{sel}_C(S))\big)=\int_T\overline{\mathrm{co}}\,\mathbb{E}(z,d|S)\left(t,\frac{d\mu}{d|\mu|}(t)\right)d|\mu|(t),$$

where $\dfrac{d\mu}{d|\mu|}(\cdot)$ is the Radon–Nikodym derivative of μ with respect to its total variation $|\mu|$.

3. Main Results

Let us first introduce the following notion. If $\widehat{w}:[0,1]\to\mathbb{R}^n$ is an arbitrary function, then the *ε-tube* around $\widehat{w}$ is the set

$$\mathcal{T}_\varepsilon(\widehat{w}):=\{(t,w)\in[0,1]\times\mathbb{R}^n \mid |w-\widehat{w}(t)|<\varepsilon \text{ for } t\in[0,1]\}.$$

Also, for $w_0\in\mathbb{R}^n$, $B_\varepsilon(w_0)$ stands for the open ball centered at w_0 of radius ε.

Let $\varepsilon>0$ be given. Let ℓ, and b be functions defined on $B_\varepsilon\big(\widehat{x}(0)\big)\times B_\varepsilon\big(\widehat{x}(1)\big)$ to $\mathbb{R}$ and $\mathbb{R}^s$, respectively. Let f be defined on $\mathcal{T}_\varepsilon(\widehat{x})\times\mathbb{R}^m$ to $\mathbb{R}^n$ and k be defined on $\mathcal{T}_\varepsilon(\widehat{x})$ to $\mathbb{R}^\kappa$. Let S be given convex subsets of $\mathbb{R}^\kappa$ with nonempty interior and U be an arbitrary subset of $\mathbb{R}^m$. The optimal control problem considered here is the problem $(\mathcal{P})$ introduced in Section 1.

The Hamiltonian function associated to $(\mathcal{P})$ is

$$\mathcal{H}(t,x,u,p):=p^T f(t,x,u)$$

A process $(\widehat{x},\widehat{u})$ is called *regular* for $(\mathcal{P})$ if the following conditions are satisfied:

(R_1) The functions ℓ,b and are strictly Fréchet differentiable at the point $(\widehat{x}(0),\widehat{x}(1))$.

(R_2) The function f is continuous on $\mathcal{T}_\varepsilon(\widehat{x})\times U$ and, for almost all $t\in[0,1]$, the map

$$(s,x)\mapsto f(s,x,\widehat{u}(t)) \qquad \big((s,x)\in\mathcal{T}_\varepsilon(\widehat{x})\big) \tag{3.1}$$

is strictly Fréchet differentiable at $(t,\widehat{x}(t))$ $\mathcal{L}^1$-uniformly for $t\in[0,1]$, and, there exists a number $\delta_0>0$ such that, for almost all $t\in[0,1]$ and $u\in U$, the map

$$(s,x)\mapsto f(s,x,u) \tag{3.2}$$

is Lipschitz on $B_{\delta_0}(t,\widehat{x}(t))\cap\mathcal{T}_\varepsilon(\widehat{x})$. Furthermore, it is also assumed that $\widehat{f}$, $\widehat{f}_t$, and $\widehat{f}_x$ are integrable functions.

(R_3) The function k defined on $\mathcal{T}_\varepsilon(\widehat{x})$ is continuous, and, for all $t\in[0,1]$, the map

$$(s,x)\mapsto k(s,x) \qquad \big((s,x)\in\mathcal{T}_\varepsilon(\widehat{x})\big) \tag{3.3}$$

is strictly Fréchet differentiable at the point $(t,\widehat{x}(t))$ uniformly in $t\in[0,1]$. Furthermore, it is also assumed that $\widehat{k}_t$ and $\widehat{k}_x$ are continuous functions.

(R_4) The sets $R\subset\mathbb{R}^r$, $S\subset\mathbb{R}^s$ are closed convex and have nonempty interior.

A pair $(\delta\theta,\delta x)$ is said to be *critical* for $(\mathcal{P})$ at $(\widehat{x},\widehat{u})$ if $\delta\theta:[0,1]\to\mathbb{R}$ is Lipschitz continuous and $\delta x:[0,1]\to\mathbb{R}^n$ is absolutely continuous, furthermore,

(C_1) $\widehat{\ell}_{x_0}\delta x(0)+\widehat{\ell}_{x_1}\delta x(1)\leqslant 0;$

(C_2) $\widehat{b}_{x_0}\delta x(0)+\widehat{b}_{x_1}\delta x(1)=0$;
(C_3) $\delta\theta(0)=\delta\theta(1)=0$;
(C_4) $\dot{\delta x}(t)=\widehat{f}_x(t)\delta x(t)+\widehat{f}_t(t)\delta\theta(t)+\widehat{f}(t)\dot{\delta\theta}(t)$ holds for a.e. $t\in[0,1]$.
(C_5) $\widehat{k}_t(t)\delta\theta(t)+\widehat{k}_x(t)\delta x(t)\in C(\widehat{k}(t)|S)$ uniformly in $t\in[0,1]$, that is, there exists a constant $M>0$ such that, for all $t\in[0,1]$,

$$\left[\zeta^T(\widehat{k}_t(t)\delta\theta(t)+\widehat{k}_x(t)\delta x(t))\right]^2\leqslant M|\zeta|\left(\delta^*(\zeta|S)-\zeta^T\widehat{k}(t)\right)$$

whenever $\zeta\in\mathbb{R}^\kappa$ satisfies $\zeta^T(\widehat{k}_t(t)\delta\theta(t)+\widehat{k}_x(t))\delta x(t)>0$.

A critical pair $(\delta\theta,\delta x)$ is called *regular* for $(\mathscr{P})$ at $(\widehat{x},\widehat{u})$ if

(R_5) the functions ℓ and b are twice directionally differentiable at $(\widehat{x}(0),\widehat{x}(1))$ in the direction $(\delta x(0),\delta x(1))$;
(R_6) for all $u\in U$, the map in (3.1) is twice directionally differentiable at $(t,\widehat{x}(t))$ in direction $(\delta\theta(t),\delta x(t))$ $\mathscr{L}^1$-uniformly in $t\in[0,1]$;
(R_7) for all $t\in[0,1]$, the map (3.3) is twice directionally differentiable at $(t,\widehat{x}(t))$ in direction $(\delta\theta(t),\delta x(t))$ uniformly in $t\in[0,1]$.

The following result offers first and second-order conditions for strong local optimality in $(\mathscr{P})$. Its proof makes use of all the results of Sections 2.

THEOREM 3.1. *Let $(\widehat{x},\widehat{u})$ be a regular strong local minimum point for the problem $(\mathscr{P})$. Then, for every regular critical pair $(\delta\theta,\delta x)$, there exist constants $\lambda\in\mathbb{R}$, $\beta=(\beta_1,\dots,\beta_s)\in\mathbb{R}^s$, an absolutely continuous function $p:[0,1]\to\mathbb{R}^n$, and a Borel regular vector-valued measure $\mu=(\mu_1,\dots,\mu_\kappa)$ not all zero, such that $\lambda\geqslant 0$,*

$$\frac{d\mu}{d|\mu|}(t)\in N(\widehat{k}(t)|S(t))\qquad \textit{for } \mu-\textit{ a.e. } t\in[0,1], \tag{3.4}$$

$$\dot{p}^T(t)=-\widehat{\mathscr{H}}_x\Big(t,p(t)+\int_{]t,1]}\widehat{k}_x^T(s)d\mu(s)\Big)\qquad \textit{for a.e. } t\in[0,1], \tag{3.5}$$

$$-p^T(0)=\lambda\widehat{\ell}_{x_0}+\beta^T\widehat{b}_{x_0}+\Big(\int_{[0,1]}\widehat{k}_x^T(t)d\mu(t)\Big)^T, \tag{3.6}$$

$$p^T(1)=\lambda\widehat{\ell}_{x_1}+\beta^T\widehat{b}_{x_1}, \tag{3.7}$$

$$\begin{aligned}&\min_{u\in U}\mathscr{H}\Big(t,\widehat{x}(t),u,p(t)+\int_{]t,1]}\widehat{k}_x^T(s)d\mu(s)\Big)\\ &=\widehat{\mathscr{H}}\Big(t,p(t)+\int_{]t,1]}\widehat{k}_x^T(s)d\mu(s)\Big)\ \textit{ for a.e. } t\in[0,1],\end{aligned} \tag{3.8}$$

and

$$
\begin{aligned}
&\big(\lambda\widehat{\ell}'' + \beta^T\widehat{b}''\big)(\delta x(0), \delta x(1)) + \int_{[0,1]} \widehat{k}''(t;\delta x(t))d\mu(t) \\
&\quad + \int_0^1 \widehat{\mathscr{H}}''\Big(t, p(t) + \int_{]t,1]} \widehat{k}_x^T(s)d\mu(s), ; \delta\theta(t), \delta x(t)\Big)dt \\
&\quad + \int_0^1 2\dot{\delta\theta}(t)\widehat{\mathscr{H}}'\Big(t, p(t) + \int_{]t,1]} \widehat{k}_x^T(s)d\mu(s), ; \delta\theta(t), \delta x(t)\Big)dt \\
&\geqslant 2\int_{[0,1]} \overline{\mathrm{co}}\,\mathbb{E}(\widehat{k}, \widehat{k}_t\delta\theta + \widehat{k}_x\delta x|S)\Big(t, \frac{d\mu}{d|\mu|}(t)\Big)d|\mu|(t),
\end{aligned}
\tag{3.9}
$$

where $\mathscr{H}'$ and $\mathscr{H}''$ denote, respectively, the first-order Fréchet derivative and the second-order directional derivative of $\mathscr{H}$ with respect to the variables (t,x).

Proof. The proof of the above theorem is based on a modification of the time transformation trick due to Dubovitskii and Milyutin.

Let $(\widehat{x}, \widehat{u})$ be a strong local minimum for $(\mathscr{P})$. Denote by T the set of full measure defined as

$$
T := \{t \in (0,1) \mid \dot{\widehat{x}}(t) = f(t, \widehat{x}(t), \widehat{u}(t)),\ \widehat{u}(t) \in U, \\
\text{and the conditions in } (R_2) \text{ hold at } t\}
$$

and choose dense subsets $\{t_1, t_2, \ldots\}$ of T and $\{u_1, u_2, \ldots\}$ of U. denote by M_{ik} the Lipschitz constant of the function $(t,x) \mapsto f(t,x,u_i)$ on $B_{\delta_0}(t_k, \widehat{x}(t_k)) \cap \mathscr{T}_\varepsilon(\widehat{x})$ and define the constants c_{ik} by $c_{ik} := |f(t_k, \widehat{x}(t_k), u_i)| + M_{ik}$ for $i,k \in \mathbb{N}$.

By Lemma 2.3 of [12], to this choice of the $t_i's$ and of the $c_{ik}'s$, there correspond a sequence of pairwise disjoint intervals $I_k = (\tau_k, \tau_k'] \subset [0,1]$ whose total length is $1/2$, functions $\widehat{t}, \widehat{v} : [0,1] \to \mathbb{R}$ defined by

$$
\widehat{t}(\tau) := \int_0^\tau \widehat{v}(s)ds, \text{ where } \widehat{v}(\tau) := \begin{cases} 2 \text{ if } \tau \notin \bigcup_{k=1}^\infty I_k, \\ 0 \text{ if } \tau \in \bigcup_{k=1}^\infty I_k, \end{cases} \qquad (\tau \in [0,1]),
$$

and satisfies $\widehat{t}(\tau) = t_k$ on I_k. Furthermore, there exists a family of pairwise disjoint measurable sets $\{A_{ik} \mid i,k \in \mathbb{N}\}$ such that the sets $A_{1k}, A_{2k}, \ldots$ form a partition of I_k for all k, and, for any open set $O \subset I_k$, we have $\operatorname{meas}(O \cap A_{ik}) > 0$ for all $i,k \in \mathbb{N}$, and $\sum_{i=1}^\infty \sum_{k=1}^\infty c_{ik} \operatorname{meas}(A_{ik}) < \infty$.

Define the functions $\widehat{y} : [0,1] \to \mathbb{R}^n$ and $\widehat{w} : [0,1] \to U$ by

$$
\widehat{y}(\tau) := \widehat{x}(\widehat{t}(\tau)), \qquad \widehat{w}(\tau) := \begin{cases} \widehat{u}(\widehat{t}(\tau)) & \text{if } \tau \notin \bigcup_{k=1}^\infty I_k, \\ u_i & \text{if } \tau \in A_{ik}. \end{cases}
$$

Now consider the following *auxilliary* control problem.

$(\mathscr{AP})$ Minimize $\ell(y(0), y(1))$ subject to

$$\dot{t}(\tau) = v(\tau), \text{ (a.e. } \tau \in [0,1]), \tag{3.10}$$

$$\dot{y}(\tau) = v(\tau) f(t(\tau), y(\tau), \widehat{w}(\tau)), \text{ (a.e. } \tau \in [0,1]), \tag{3.11}$$

$$t(0) = 0,\ t(1) = 1, \tag{3.12}$$

$$b(y(0), y(1)) = 0, \tag{3.13}$$

$$k(t(\tau), y(\tau)) \in S,\ (\tau \in [0,1]), \tag{3.14}$$

$$v(\tau) \geqslant 0, \text{ (a.e. } \tau \in [0,1]), \tag{3.15}$$

where t and y are absolutely continuous functions and v is an essentially bounded measurable function over $[0, 1]$.

The Hamiltonian corresponding to $(\mathscr{AP})$ is

$$H(\tau, t, y, v, r, q, \rho) := vq^T f(t, y, \widehat{w}(\tau)) + rv - \rho v.$$

Similarly to [3], one can show that if $(\widehat{x}, \widehat{u})$ is a strong local minimum for $(\mathscr{P})$ then $(\widehat{t}, \widehat{y}, \widehat{v})$ is a strong local minimum for $(\mathscr{AP})$. In particular, $(\widehat{t}, \widehat{y}, \widehat{v})$ is a *weak* local minimum for $(\mathscr{AP})$.

For $z = (t, y) \in \mathbb{R}^{n+1}$, $v \in \mathbb{R}$, and $\tau \in [0, 1]$, set

$$\boldsymbol{f}(\tau, z, v) := \begin{pmatrix} v \\ vf(t, y, \widehat{w}(\tau)) \end{pmatrix}, \quad \boldsymbol{k}(\tau, z) := k(t, y),$$

$$\boldsymbol{\mathscr{H}}(\tau, z, v, \zeta, \rho) := \zeta^T \boldsymbol{f}(\tau, z, v) - \rho v.$$

By using the above defined functions, it is readily seen that $(\mathscr{AP})$ is in the form of $(\mathscr{P})$, where f is replaced by $\boldsymbol{f}$. Hence, in order to apply Theorem 4.1 of [11] to $(\mathscr{AP})$, we need to verify that $(\widehat{t}, \widehat{y}, \widehat{v})$ is a regular solution for $(\mathscr{AP})$. Furthermore, we need to construct the regular critical directions for $(\mathscr{AP})$ in terms of those for $(\mathscr{P})$. The fruits of the investigation conducted in [12] is stated below.

PROPOSITION 3.1. *If $(\widehat{x}, \widehat{u})$ is a regular solution for problem $(\mathscr{P})$, then $(\widehat{t}, \widehat{y}, \widehat{v})$ is also a regular solution for $(\mathscr{AP})$.*

PROPOSITION 3.2. *Let $(\delta\theta, \delta x)$ be a regular critical direction for $(\mathscr{P})$. Then $(\delta t, \delta x, \delta v)$ defined by*

$$\delta t(\tau) := \delta\theta(\widehat{t}(\tau)),\ \delta y(\tau) := \delta x(\widehat{t}(\tau)),\ \delta v(\tau) := \widehat{v}(\tau)\dot{\delta\theta}(\widehat{t}(\tau)), \quad (\tau \in [0,1]). \tag{3.16}$$

is a regular critical direction for the problem $(\mathscr{AP})$, i.e, $(\delta z, \delta v)$ is a regular and critical direction for $(\mathscr{P})$, where $\delta z := (\delta t, \delta y)$.

Now, let $(\delta\theta, \delta x)$ be a regular critical direction for $(\mathscr{P})$ and define $(\delta t, \delta y, \delta v)$ by (3.16). Then, by Proposition 3.2, $(\delta t, \delta y, \delta v)$ is a regular critical direction for $(\mathscr{AP})$. Hence, by Theorem 4.1 in [11] applied to $(\mathscr{AP})$, there exist $\lambda \in \mathbb{R}$, $\boldsymbol{\beta} = (\beta, \gamma_0, \gamma_1) \in \mathbb{R}^{s+2}$, an absolutely continuous functions $\zeta =: (r, q) : [0,1] \to \mathbb{R} \times \mathbb{R}^n$, an integrable function $\rho : [0,1] \to \mathbb{R}$, and a Borel regular vector valued measure $\boldsymbol{\mu}$ not all zero such that $\lambda \geqslant 0$ and

$$\rho(\tau) \geqslant 0 \quad \text{and} \quad \rho(\tau)\widehat{v}(\tau) = 0 \quad \text{for a.e. } \tau \in [0,1], \tag{3.17}$$

$$\frac{d\boldsymbol{\mu}}{d|\boldsymbol{\mu}|}(\tau) \in N(\widehat{\boldsymbol{k}}(\tau)|S) \quad \text{for } \mu- \text{ a.e. } \tau \in [0,1], \tag{3.18}$$

$$\dot{\zeta}^T(\tau) = -\widehat{\mathscr{H}}_z\Big(\tau, \zeta(\tau) + \int_{]\tau,1]} \widehat{\boldsymbol{k}}_z^T(\sigma)\, d\boldsymbol{\mu}(\sigma), \rho(\tau)\Big) \quad \text{for a.e. } \tau \in [0,1], \tag{3.19}$$

$$-\zeta^T(0) = \lambda \widehat{\ell}_{z_0} + \beta^T \widehat{b}_{z_0} + \Big(\int_{[0,1]} \widehat{\boldsymbol{k}}_z^T(\tau)\, d\boldsymbol{\mu}(\tau)\Big)^T, \tag{3.20}$$

$$\zeta^T(1) = \lambda \widehat{\ell}_{z_1} + \beta^T \widehat{b}_{z_1}, \tag{3.21}$$

$$\widehat{\mathscr{H}}_v\Big(\tau, \zeta(\tau) + \int_{]\tau,1]} \widehat{\boldsymbol{k}}_z^T(\sigma)\, d\boldsymbol{\mu}(\sigma), \rho(\tau)\Big) = 0 \quad \text{for a.e. } \tau \in [0,1], \tag{3.22}$$

and

$$\begin{aligned} &\big(\lambda \widehat{\ell}'' + \beta^T \widehat{b}''\big)(\delta z(0), \delta z(1)) + \int_{[0,1]} \widehat{\boldsymbol{k}}''(\tau; \delta z(\tau))\, d\boldsymbol{\mu}(\tau) \\ &+ \int_0^1 \widehat{\mathscr{H}}''\Big(\tau, \zeta(\tau) + \int_{]\tau,1]} \widehat{\boldsymbol{k}}_z^T(\sigma)\, d\boldsymbol{\mu}(\sigma), \rho(\tau); \delta z(\tau), \delta v(\tau)\Big) d\tau \\ &\geqslant 2 \int_{[0,1]} \overline{\operatorname{co}}\, \mathbb{E}(\widehat{\boldsymbol{k}}, \widehat{\boldsymbol{k}}_z \delta z | S)\Big(\tau, \frac{d\boldsymbol{\mu}}{d|\boldsymbol{\mu}|}(\tau)\Big) d|\boldsymbol{\mu}|(\tau), \end{aligned} \tag{3.23}$$

where $\mathscr{H}''$ denotes the second-order strong directional derivative of $\mathscr{H}$ with respect to the variable (z, v).

In order to be able to translate condition (3.18) into (3.4), we need the following result from [12].

Set

$$\widehat{\tau}(s) := \begin{cases} \min\{\sigma \in [0,1] \,|\, \widehat{t}(\sigma) = s\} & \text{if } s < 1, \\ 1 & \text{if } s = 1. \end{cases}$$

Then, $\widehat{\tau}$ is strictly increasing and hence, is differentiable at almost all s. Moreover, $\widehat{t}(\tau(s)) = s$, for all $s \in [0,1]$.

PROPOSITION 3.3. *Let $\boldsymbol{\mu}$ be a finite Borel measure in $\big(\mathscr{C}([0,1],\mathbb{R}^\kappa)\big)^*$ and let μ be defined by*

$$\mu(Q)=\boldsymbol{\mu}\big(\widehat{\tau}(Q)\big)+\sum_{t_k\in Q}\boldsymbol{\mu}(I_k), \tag{3.24}$$

where Q is a Borel subset of $[0,1]$. *Then, for any bounded Borel measurable function $\varphi:[0,1]\to\mathbb{R}^\kappa$,*

$$\int_{[0,1]}\varphi(\widehat{t}(\tau))d\boldsymbol{\mu}(\tau)=\int_{[0,1]}\varphi(t)d\mu(t). \tag{3.25}$$

Furthermore, if S is closed convex subset of $\mathbb{R}^\kappa$ with nonempty interior and K: $[0,1]\to S$ is a continuous function, then

$$\frac{d\boldsymbol{\mu}}{d|\boldsymbol{\mu}|}(\tau)\in N\big(K(\widehat{t}(\tau))\big|S\big) \qquad \text{for } \boldsymbol{\mu} \text{ a.e. } \tau\in[0,1]$$

if and only if

$$\frac{d\mu}{|d\mu|}(t)\in N\big(K(t)\big|S\big) \qquad \text{for } \mu\text{-a.e. } t\in[0,1]$$

and, for all $k\in\mathbb{N}$ and for all Borel sets $W\subset\bar{I}_k$

$$\boldsymbol{\mu}(W)\in N\big(K(t_k)\big|S\big).$$

Applying the above proposition with $K(t):=k(t,\widehat{x}(t))$, we can see that (3.18) implies (3.4) if the measure μ is defined by (3.24).

Now we continue or analysis with the adjoint equation (3.19) which is equivalent to

$$\begin{aligned}\dot{r}(\tau) &= -\widehat{v}(\tau)f_t^T\big(\widehat{t}(\tau),\widehat{y}(\tau),\widehat{w}(\tau)\big)\\ &\qquad\Big(q(\tau)+\int_{(\tau,1]}k_x^T(\widehat{t}(\sigma),\widehat{y}(\sigma))d\boldsymbol{\mu}(\sigma)\Big)\end{aligned} \tag{3.26}$$

$$\begin{aligned}\dot{q}(\tau) &= -\widehat{v}(\tau)f_x^T\big(\widehat{t}(\tau),\widehat{y}(\tau),\widehat{w}(\tau)\big)\\ &\qquad\Big(q(\tau)+\int_{(\tau,1]}k_x^T(\widehat{t}(\sigma),\widehat{y}(\sigma))d\boldsymbol{\mu}(\sigma)\Big)\end{aligned} \tag{3.27}$$

Observe that q and r are constant on I_k for all k. Set, for $t\in[0,1]$,

$$p(t):=q(\widehat{\tau}(t)), \qquad \psi(t):=r(\widehat{\tau}(t)). \tag{3.28}$$

Then these functions are absolutely continuous and, for a.e. $t\in[0,1]$,

$$\dot{p}(t)=\frac{\dot{q}(\widehat{\tau}(t))}{\widehat{v}(\widehat{\tau}(t))}, \qquad \dot{\psi}(t)=\frac{\dot{r}(\widehat{\tau}(t))}{\widehat{v}(\widehat{\tau}(t))}. \tag{3.29}$$

(for the proof, see the argument followed in [3]).

Observe that, for $t\in[0,1]$, the formula (3.25) applied to the function $\varphi(s):=\chi_{(t,1]}(s)K(s,\widehat{x}(s))$ and the identity $\chi_{(t,1]}(\widehat{t}(\sigma))=\chi_{(\widehat{\tau}(t),1]}(\sigma)$ yield that

$$\int_{(\widehat{\tau}(t),1]} K^T(\widehat{t}(\sigma),\widehat{y}(\sigma))d\boldsymbol{\mu}(\sigma)=\int_{(t,1]} K^T(s,\widehat{x}(s))d\mu(s)$$

for $K=k_t$ and $K=k_x$. Thus, substituting $\tau=\widehat{\tau}(t)$ into (3.26) and (3.27) and using (3.28), (3.29), we get that

$$\dot{\psi}(t)=-\widehat{f}_t^T(t)\Big(p(t)+\int_{(t,1]}\widehat{k}_x^T(s)d\mu(s)\Big)$$

$$\dot{p}(t)=-\widehat{f}_x^T(t)\Big(p(t)+\int_{(t,1]}\widehat{k}_x^T(s)d\mu(s)\Big)$$

hold for a.e. $t\in[0,1]$. Now, observe that the latter equation is equivalent to (3.5).

The conditions (3.20) and (3.21) directly yield (again by using (3.25)) that (3.6) and (3.7) are satisfied. For the function ψ, we obtain the following endpoint conditions

$$-\psi(0)=\gamma_0+\int_0^1\widehat{k}_t^T(t)d\mu(t),\qquad \psi(1)=\gamma_1.$$

By equation (3.22), we have, for a.e. $\tau\in[0,1]$,

$$f^T\big(\widehat{t}(\tau),\widehat{y}(\tau),\widehat{w}(\tau)\big)\Big(q(\tau)+\int_{(\tau,1]}k_x^T(\widehat{t}(\sigma),\widehat{y}(\sigma))d\boldsymbol{\mu}(\sigma)\Big)$$
$$+\Big(r(\tau)+\int_{(\tau,1]}k_t^T(\widehat{t}(\sigma),\widehat{y}(\sigma))d\boldsymbol{\mu}(\sigma)\Big)=\rho(\tau). \tag{3.30}$$

Now we distinguish two cases. By the second condition in (3.17), $\rho(\tau)=0$ for a.e. $\tau\in C$. Therefore, with the substitution $\widehat{\tau}(t)$, it follows from the above equation that

$$\widehat{f}^T(t)\Big(p(t)+\int_{(t,1]}\widehat{k}_x^T(s)d\mu(s)\Big)+\Big(\psi(t)+\int_{(t,1]}\widehat{k}_t^T(s)d\mu(s)\Big)=0 \tag{3.31}$$

for a.e. $t\in[0,1]$.

Let now $k\in\mathbb{N}$ be fixed. Then, by the first condition in (3.17), $\rho(\tau)\geqslant 0$ for a.e. $\tau\in I_k$. Therefore, it follows from (3.30) that

$$f^T\big(\widehat{t}(\tau),\widehat{y}(\tau),\widehat{w}(\tau)\big)\Big(q(\tau)+\int_{(\tau,1]}k_x^T(\widehat{t}(\sigma),\widehat{y}(\sigma))d\boldsymbol{\mu}(\sigma)\Big)$$
$$+\Big(r(\tau)+\int_{(\tau,1]}k_t^T(\widehat{t}(\sigma),\widehat{y}(\sigma))d\boldsymbol{\mu}(\sigma)\Big)\geqslant 0 \tag{3.32}$$

for a.e. $\tau\in I_k$. Since, for all i, A_{ik} is intersecting any open subset of I_k at a set of positive measure, therefore, for each i, there is a sequence $\sigma_n\in A_{ik}$ tending to τ_k

such that (3.32) holds at $\tau=\sigma_n$. Putting $\tau=\sigma_n$ into (3.32) and upon taking the limit, we arrive at

$$f^T\big(t_k,\widehat{x}(t_k),u_i\big)\Big(p(t_k)+\int_{(t_k,1]}\widehat{k}_x^T(s)d\mu(s)\Big)+\Big(\psi(t_k)+\int_{(t_k,1]}\widehat{k}_t^T(s)d\mu(s)\Big)\geqslant 0, \tag{3.33}$$

where we utilized that $\widehat{t}(\sigma_n)=t_k$, $\widehat{y}(\sigma_n)=t_k$, $\widehat{w}(\sigma_n)=u_i$ hold for all n. Furthermore,

$$\begin{aligned}\lim_{n\to\infty}\int_{(\sigma_n,1]}K^T(\widehat{t}(\sigma),\widehat{y}(\sigma))d\boldsymbol{\mu}(\sigma)&=\int_{(\tau_k,1]}K^T(\widehat{t}(\sigma),\widehat{y}(\sigma))d\boldsymbol{\mu}(\sigma)\\&=\int_{(t_k,1]}\widehat{K}^T(s)d\mu(s).\end{aligned}$$

Since f is continuous in u and $\{u_1,u_2,\dots\}$ is dense in U, (3.33) yields that

$$f^T\big(t_k,\widehat{x}(t_k),u\big)\Big(p(t_k)+\int_{(t_k,1]}\widehat{k}_x^T(s)d\mu(s)\Big)+\Big(\psi(t_k)+\int_{(t_k,1]}\widehat{k}_t^T(s)d\mu(s)\Big)\geqslant 0 \tag{3.34}$$

is satisfied for all $u\in U$ and for all $k\in\mathbb{N}$. A similar argument (that uses the continuity of $f(\cdot,\cdot,u)$) and the fact that $\{t_1,t_2,\dots\}$ is dense in $(0,1)$) shows that (3.34) holds when t_k is replaced by any $t\in[0,1)$. The resulting inequality combined with (3.31) yields that, for a.e. $t\in[0,1]$ and for all $u\in U$,

$$f^T\big(t,\widehat{x}(t),u\big)\Big(p(t)+\int_{(t,1]}\widehat{k}_x^T(s)d\mu(s)\Big)\geqslant\widehat{f}^T(t)\Big(p(t)+\int_{(t,1]}\widehat{k}_x^T(s)d\mu(s)\Big).$$

Thus, we have proved that (3.8) is satisfied.

The fact that some of the multipliers $\lambda,\boldsymbol{\beta},p,\mu$ has to be different from zero was established in [12].

The last part of the proof consists of analyzing (3.23) in order to obtain (3.9). Note that the first two terms in (3.9) are direct translations of the corresponding terms in (3.23) where (3.25) was employed. For the third term, observe that

$$\widehat{\mathcal{H}}''(\tau,z,v,\zeta,\psi;\delta z,\delta v)=\zeta^T\widehat{\boldsymbol{f}}''(\tau;\delta z,\delta v)$$

and

$$\begin{aligned}&\widehat{\boldsymbol{f}}''(\tau;\delta z(\tau),\delta v(\tau)))\\&=\begin{pmatrix}0\\ \widehat{v}(\tau)\widehat{f}''(\widehat{t}(\tau);\delta t(\tau),\delta y(\tau))+2\delta v(\tau)\big[\widehat{f}_t(\widehat{t}(\tau))\delta t(\tau)+\widehat{f}_x(\widehat{t}(\tau))\delta y(\tau)\big]\end{pmatrix}.\end{aligned}$$

Whence, by (3.25), the third term of (3.23) transforms to the third and fourth terms of (3.9).

In order to compare the last terms of these inequalities, we shall need the following result.

LEMMA 1. *Using the notations introduced earlier, we have*

$$\delta^*\Big(\boldsymbol{\mu}\Big|V\big(\widehat{\boldsymbol{k}},\widehat{\boldsymbol{k}}_t\delta t+\widehat{\boldsymbol{k}}_x\delta y\big|\mathrm{sel}_C(S)\big)\Big)\geqslant\delta^*\Big(\mu\Big|V\big(\widehat{k},\widehat{k}_t\delta\theta+\widehat{k}_x\delta x\big|\mathrm{sel}_C(S)\big)\Big), \tag{3.35}$$

where δ^* stands for the support function.

Proof. Let $\eta_0:[0,1]\to\mathbb{R}^\kappa$ be an arbitrary second-order variation in $V\big(\widehat{k},\widehat{k}_t\delta\theta+\widehat{k}_x\delta x\big|\mathrm{sel}_C(S)\big)$. Then, there exists $\bar{\varepsilon}>0$ such that

$$\widehat{k}(t)+s\big[\widehat{k}_t(t)\delta\theta(t)+\widehat{k}_x(t)\delta x(t)\big]+s^2[\eta_0(t)+\eta]\in S \tag{3.36}$$

for all $t\in[0,1]$, for $|\eta|<\bar{\varepsilon}$, and for all $s\in(0,\bar{\varepsilon})$.

Now define $\gamma_0(\tau):=\eta_0(\widehat{t}(\tau))$ for all $\tau\in[0,1]$. By substituting $t=\widehat{t}(\tau)$ into (3.36), it results that γ_0 belongs to $V\big(\widehat{\boldsymbol{k}},\widehat{\boldsymbol{k}}_t\delta t+\widehat{\boldsymbol{k}}_x\delta y\big|\mathrm{sel}_C(S)\big)$.

Furthermore, by (3.25), we obtain

$$\int_0^1\eta_0^T(t)d\mu(t)=\int_0^1\gamma_0^T(\tau)d\boldsymbol{\mu}(\tau).$$

Therefore, the conclusion of the Lemma is proved. □

From [6, Theorem 3.10], we have the terms on the right hand side of equations (3.23) and (3.9) are, respectively, equal to the left and right hand side of inequality (3.35). Hence, (3.23) implies that (3.9) holds.

Therefore the proof of theorem 3.1 is now completed. □

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